## §6.3 Morphisms of Varieties

$$\begin{tikzcd} X \arrow[r, "\varphi \ \leftarrow \text{cont.}"] & Y \\ \varphi^{-1}(U) \arrow[u, phantom, "\subseteq" rotate=90] \arrow[r, "\varphi"] \arrow[dr, "\tilde{\varphi}(f)"'] & U \arrow[u, phantom, "\subseteq" rotate=90] \arrow[d, "f"] \\ & k \end{tikzcd} \Rightarrow \begin{tikzcd} \mathcal{F}(U, k) \arrow[r, "\tilde{\varphi}"] & \mathcal{F}(\varphi^{-1}(U), k) \\ P(U) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, dashed, "??"] & P(\varphi^{-1}(U)) \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

**Def** Let $X$ and $Y$ be varieties. A **morphism** from $X$ to $Y$ is a mapping $\varphi : X \to Y$ such that

1) $\varphi$ is continuous
2) $\forall\, U \overset{\text{o}}{\hookrightarrow} Y,\ \forall f \in \Gamma(U) \Rightarrow \tilde{\varphi}(f) := f \circ \varphi \in \Gamma(\varphi^{-1}(U))$

" locally defined by polynomial "

**Def**: An **isomorphism** of $X$ with $Y$ is a bijection $\varphi : X \to Y$ such that both $\varphi$ and $\varphi^{-1}$ are homomorphism.

**affine variety** := variety isomorphic to closed subvar. of some $\mathbb{A}^n$

**projective variety** := variety isomorphic to closed subvar. of some $\mathbb{P}^n$.

How to find if a mapping is a morphism or not. $\to$ open covering

**Prop:** $X, Y$ = varieties. $f: X \to Y$ mapping. $X = \bigcup_\alpha U_\alpha$, $Y = \bigcup_\alpha V_\alpha$.

$f_\alpha := f|_{U_\alpha} : U_\alpha \to V_\alpha$ Then

$f$ = morphism $\Leftrightarrow$ $f_\alpha$ = morphism for all $\alpha$

Pf: $\forall\ V \overset{\text{open}}{\subseteq} Y \Rightarrow f^{-1}(V) = \bigcup_\alpha f_\alpha^{-1}(V \cap V_\alpha) \overset{\text{open}}{\subseteq} X \Rightarrow f = \text{cont}$ ($f_\alpha$ = cont.)

$$\tilde{f}(\Gamma(V)) = \tilde{f}\left(\bigcap_\alpha \Gamma(V_\alpha \cap V)\right) = \bigcap_\alpha \tilde{f}_\alpha(\Gamma(V_\alpha \cap V))$$

$$\subseteq \bigcap_\alpha \Gamma(U_\alpha \cap f^{-1}(V)) = \Gamma(f^{-1}(V))$$

Which ones can we determine?:

**Prop 2.** $\{\varphi: X \to Y \mid \text{morphism}\} \xleftrightarrow[1:1]{X, Y = \text{affine}} \{\tilde{\varphi}: \Gamma(Y) \to \Gamma(X) \mid \text{ring hom}\}$

Pf: WMA: $X \overset{\text{closed}}{\subseteq} \mathbb{A}^n$, $Y \overset{\text{closed}}{\subseteq} \mathbb{A}^m$

$$\{\tilde{\varphi}: \Gamma(Y) \to \Gamma(X) \mid \text{ring hom}\} \underset{\text{Prop 1 §2.2.}}{\overset{1:1}{\longleftrightarrow}} \{\varphi: X \to Y \mid \text{poly. map}\}$$

$$\cup|$$

$$\{\varphi: X \to Y \mid \text{morphism}\}$$

ONTS: $\forall$ poly map is a morphism

$\forall\ h \in \Gamma(U)\ \forall\ Q \in \varphi^{-1}(U)$, $p := \varphi(Q) \in U$

$h \overset{\exists}{=} \frac{f}{g} \in \Gamma(U)$ $\begin{cases} f = F \bmod I(Y) \in \Gamma(Y) \\ g = G \bmod I(Y) \in \Gamma(Y) \end{cases}$ s.t. $g(p) \neq 0$.

$\varphi = \text{poly} \Rightarrow \tilde{\varphi}(f), \tilde{\varphi}(g) \in \Gamma(X) \Rightarrow \tilde{\varphi}(h) = \frac{\tilde{\varphi}(f)}{\tilde{\varphi}(g)} \in k(X)$

$\tilde{\varphi}(g)|_Q = g|_{\varphi(Q)} = g(p) \neq 0 \Rightarrow \tilde{\varphi}(h)$ is defined at $Q$.

$\Rightarrow \tilde{\varphi}(h) \in \Gamma(\varphi^{-1}(U)) \Rightarrow \checkmark$.

Can we cover any varieties with affine subvarieties?

**Prop 3.** proj var. is a union of finite open affine varities.

Pf: $V \hookrightarrow \mathbb{P}^n$. $\varphi_i : \mathbb{A}^n \xrightarrow{\cong} U_i \hookrightarrow \mathbb{P}^n$.

$V_i := \varphi_i^{-1}(V) \hookrightarrow \mathbb{A}^n$ Then

1) $\varphi_i|_{V_i} : V_i \xrightarrow{\cong} V \cap U_i$

2) $V = \bigcup_i (V \cap U_i)$

**Prop 4.** 1) Any closed subvariety of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is a projective variety.

2). Any variety is isomorphic to an open subvariety of a projective variety.

Pf: 1) $\Rightarrow$ 2) : clear

1): ONTS: $\mathbb{P}^m \times \mathbb{P}^n$ is a proj var.

Prob 4.28 $\Rightarrow$ Segre imbedding

$$S : \mathbb{P}^m \times \mathbb{P}^n \xrightarrow{1:1} V \hookrightarrow \mathbb{P}^{mn+m+n}$$

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \longmapsto (x_0 y_0 : \cdots : x_0 y_n : \cdots : x_m y_0 : \cdots : x_m y_n)$$

ONTS: $S|_{U_0 \times U_0} : U_0 \times U_0 \xrightarrow{\cong} V \cap V_{00}$

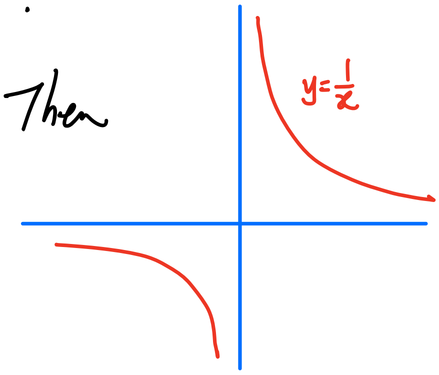
ONTS : $\Gamma(U_0 \times U_0) \xrightarrow{\cong} \Gamma(V \cap V_{00})$

$$\Gamma(U_0 \times U_0) = k[x_1 \cdots x_m, y_1 \cdots y_n]$$

$$\Gamma(V \cap V_{00}) = k[T_{10}, \cdots, T_{mn}] / (\{T_{jk} - T_{j0} T_{0k}\}_{j,k})$$

**Prop 3.** $V$ = affine var. $f \in \Gamma(V) \setminus \{0\}$.

$V_f := \{ P \in V \mid f(P) \neq 0 \}$ Then




1) $V_f \overset{o}{\hookrightarrow} V$

2) $\Gamma(V_f) = \Gamma(V)[\frac{1}{f}]$

3). $V_f$ = affine var.

Pf: WMA: $V \subset \mathbb{A}^n$, $I = I(V) \triangleleft k[X_1 \cdots X_n]$

$\Gamma(V) = k[X_1, \cdots, X_n]/I$, $f = F \bmod I$

1). $V_f = V \cap \{ P \in \mathbb{A}^n \mid F(P) \neq 0 \} \overset{o}{\hookrightarrow} V$

2). $\forall z \in \Gamma(V_f)$.

$J = \{ G \in k[X_1 \cdots X_n] \mid \overline{G} z \in \Gamma(V) \} \supseteq I(V)$

pf of prop 2 §2.4 $\Rightarrow V(J) =$ pole set of $z \subseteq V$

$\xRightarrow{z \in \Gamma(V_f)} V(J) \subseteq V(F)$

$\Rightarrow F^N \subseteq J$ for some $N$ $\Rightarrow f^N z =: a \in \Gamma(V)$

$\Rightarrow z = \frac{a}{f^N} \in \Gamma(V)[\frac{1}{f}]$

3). $I' := (I, X_{n+1}F - 1) \triangleleft k[X_1, \cdots, X_{n+1}]$

$V' := V(I') \hookrightarrow \mathbb{A}^{n+1}$

$\alpha : k[X_1, \cdots, X_{n+1}] \twoheadrightarrow \Gamma(V_f)$ $\quad x_i \longmapsto \overline{x_i} \quad (1 \le i \le n)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad X_{n+1} \longmapsto \frac{1}{f}$

Prob 6.13 $\Rightarrow R[X]/(Xf-1) \xrightarrow{\cong} R[\frac{1}{f}] \quad \overline{x} \longleftarrow\!\mapsfrom \frac{1}{f} \quad x \mapsto \frac{1}{f}$

$\Rightarrow \ker \alpha = I'$

$I' \subseteq \ker\alpha$ clear

$\Sigma := \ker\alpha \setminus I'$ Suppose $\Sigma \neq \phi$.

Let $g = r_0 + r_1 x + \cdots + r_d x^d \in \Sigma$ be of mini degree

$\Rightarrow \alpha(g) = 0 \Rightarrow f^d r_0 + \cdots + f r_{d-1} + r_d = 0$

$\Rightarrow f \mid r_d$

$\Rightarrow g' = g - \frac{r_d}{f}(xf-1) \in \Sigma$ ↯

$\alpha \Rightarrow \overline{\alpha} : \Gamma(V') \cong \Gamma(V_f) \Rightarrow \varphi : V_f \xrightarrow{\cong} V' \hookrightarrow \mathbb{A}^{n+1}$